# Discovering the Chain Rule Graphically

The chain rule is one of the hardest ideas to convey to students in Calculus I. It is difficult to motivate, so that most students do not really see where it comes from; it is difficult to express in symbols even after it is developed; and it is awkward to put it into words, so that many students can't remember it and so can't apply it correctly.

In this note, we present a way to develop the chain rule in class that both motivates it and gives the students a much better understanding of what is going on. We presume that the students have seen the notion of the derivative at a point and that they are familiar with the idea of approximating the value of that derivative via the Newton difference-quotient. We build on that approximation by using the numerical derivative routine built into most graphing calculators. For instance, on the TI-83, it takes the form nDeriv ($y1$, $x$, $a$), which approximates the value of the derivative, with respect to the variable $x$, of the function stored as $y1$ at the point $x = a$.

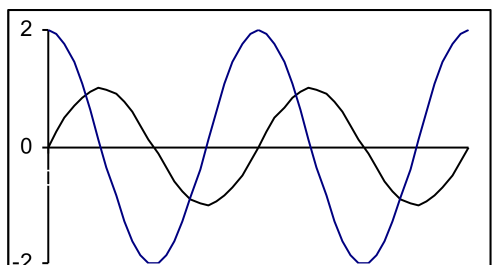

Figure 1

We begin with the function $f(x) = \sin(2x)$ and graph it on the interval $[0, 2\pi]$, along with the graph of the derivative (using the calculator command:

$$y2 = \text{nDeriv}(y1, x, x),$$

which calculates the derivative of the function $y1$ at each of the points $x$ in the designated window). The results are shown in Figure 1. The resulting derivative function looks like a cosine curve with a considerably larger amplitude than the original sine curve. With a little exploration, the students come up with the formula $f'(x) = 2\cos(2x)$. To see where the factor 2 comes from, we use the following reasoning: since $\sin(2x)$ is "moving" twice as fast as sin $x$ does, it is changing twice as fast, so the derivative must be twice as large.

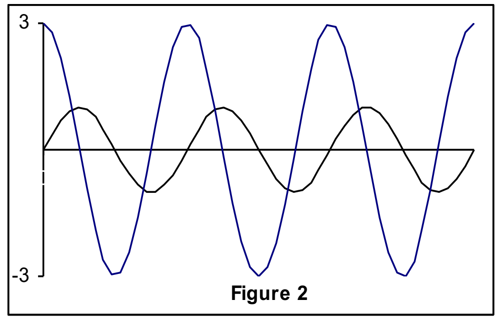

Figure 2

We then repeat this process with $f(x) = \sin(3x)$, as shown in Figure 2, to discover that the derivative is now $f'(x) = 3\cos(3x)$. Again, the reasoning is that, since $\sin(3x)$ is moving three times as fast as $\sin(x)$, its derivative must be three times as large. This quickly leads to the supposition that the derivative of $f(x) = \sin(mx)$, for any multiple $m$, must be $f'(x) = m\cos(mx)$. (This could be tied in nicely with some examples or problems on discovering the formula for the derivative of $y = e^{mx}$, for $m = 2, 3, \ldots$, based on the product rule.)

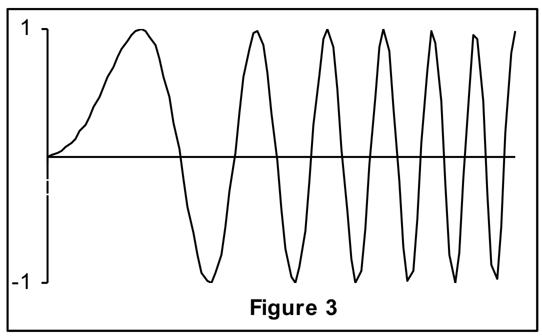

Figure 3

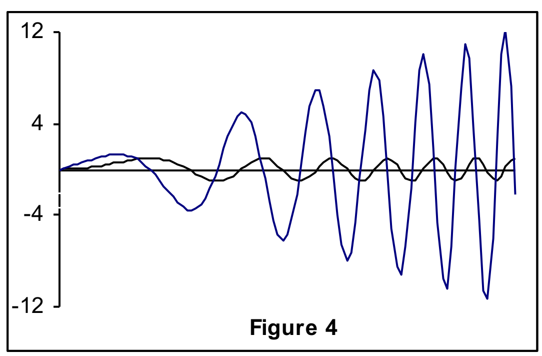

Figure 4

Next, we consider $f(x) = \sin(x^2)$, which oscillates ever more rapidly, as seen in Figure 3 on the interval $[0, 2\pi]$. The associated derivative function should therefore grow ever more rapidly. When we look at the derivative function in Figure 4, again based on using the nDeriv command, we see that this is borne out – the derivative oscillates ever more wildly, but notice that it has its roots whenever the original sine function passes its maximum or minimum points. This suggests a cosine curve with a varying amplitude, but one that somehow passes through the origin. Furthermore, observe that both the successive crests and the successive troughs of the derivative curve in Figure 4 appear to fall into linear patterns, so we can attempt to find, graphically, a pair of lines through the origin that fit these turning points. This could be done by trial and error or by tracing the derivative function to locate the maximum points, say, as precisely as possible and using the data fitting routines of a calculator to find the best linear function that fits these points. Either way, we find that $y = 2x$ seems to fit these peaks perfectly, as shown in Figure 5. Thus, the derivative seems to be equal to the cosine function with a variable amplitude equal to $2x$. This can be "verified" graphically by plotting this supposed derivative function $f'(x) = 2x \cos(x^2)$ and seeing that it fits perfectly over the approximate derivative function obtained numerically.

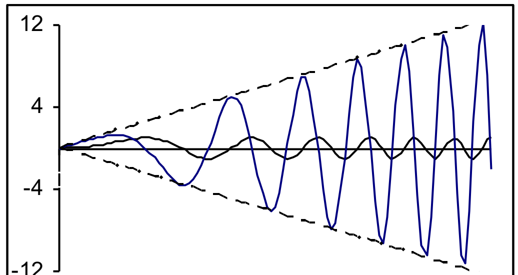

**Figure 5**

More importantly, the students see that the result involves the rate of change of the original sine function times the rate of change of the argument of that function and thus they have the chain rule! Further, this theme that the derivative of a composite function involves multiplying the rate of change of the original function by the rate of change of the argument also seems to connect very well for many students; it certainly seems to make more sense to them than talking about the derivative of the outer function times the derivative of the inner function or any of the other ways that we typically verbalize the chain rule.

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George Miller, Editor
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Dear George

I am enclosing three copies of a short article entitled *Discovering the Chain Rule Graphically* for your consideration for possible publication in MACE.

Thank you for your kind consideration. I look forward to hearing from you.

Sincerely yours,

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