

The Definite Integral as an Average

We know how to find the average of n numbers: Add them and divide by n . But how do we find the average value of a continuously varying function? Let us consider an example. Suppose $f(t)$ is the temperature at time t , measured in hours since midnight, and that we want to calculate the average temperature over a 24-hour period. One way to start is to average the temperatures at n equally spaced times, t_1, t_2, \dots, t_n , during the day.

$$\text{Average temperature} \approx \frac{f(t_1) + f(t_2) + \dots + f(t_n)}{n}.$$

The larger we make n , the better the approximation. We can rewrite this expression as a Riemann sum over the interval $0 \leq t \leq 24$ if we use the fact that $\Delta t = 24/n$, so $n = 24/\Delta t$:

$$\begin{aligned} \text{Average temperature} &\approx \frac{f(t_1) + f(t_2) + \dots + f(t_n)}{24/\Delta t} \\ &= \frac{f(t_1)\Delta t + f(t_2)\Delta t + \dots + f(t_n)\Delta t}{24} \\ &= \frac{1}{24} \sum_{i=1}^n f(t_i)\Delta t. \end{aligned}$$

As $n \rightarrow \infty$, the Riemann sum tends toward an integral, and $1/24$ of the sum also approximates the average temperature better. It makes sense, then, to write

$$\text{Average temperature} = \lim_{n \rightarrow \infty} \frac{1}{24} \sum_{i=1}^n f(t_i)\Delta t = \frac{1}{24} \int_0^{24} f(t) dt.$$

We have found a way of expressing the average temperature over an interval in terms of an integral. Generalizing for any function f , if $a < b$, we define

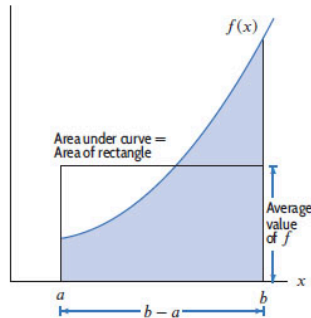
$$\text{Average value of } f \text{ from } a \text{ to } b = \frac{1}{b-a} \int_a^b f(x) dx.$$

How to Visualize the Average on a Graph

The definition of average value tells us that

$$(\text{Average value of } f) \cdot (b - a) = \int_a^b f(x) dx.$$

Let's interpret the integral as the area under the graph of f . Then the average value of f is the height of a rectangle whose base is $(b - a)$ and whose area is the same as the area under the graph of f . (See [Figure 5.77](#).)



[Figure 5.77](#) Area and average value

Example 6

Suppose that $C(t)$ represents the daily cost of heating your house, measured in dollars per day, where t is time measured in days and $t = 0$ corresponds to January 1, 2015. Interpret $\int_0^{90} C(t) dt$ and $\frac{1}{90-0} \int_0^{90} C(t) dt$.

Solution

The units for the integral $\int_0^{90} C(t) dt$ are (dollars/day) \times (days) = dollars. The integral represents the total cost in dollars to heat your house for the first 90 days of 2015, namely the months of January, February, and March. The second expression is measured in (1/days)(dollars) or dollars per day, the same units as $C(t)$. It represents the average cost per day to heat your house during the first 90 days of 2015.

[Example 5.4.6 Video: Interpreting a definite integral](#)

Example 7

In the year 2014, the population of Nevada¹³ was modeled by the function

$$P = f(t) = 2.839(1.017)^t,$$

where P is in millions of people and t is in years since 2014. Use this function to predict the average population of Nevada between the years 2010 and 2020.

Solution

We want the average value of $f(t)$ between $t = -6$ and $t = 4$. This is given by

$$\text{Average population} = \frac{1}{4 - (-6)} \int_{-6}^4 f(t) dt = \frac{1}{10}(28.91) = 2.891.$$

We used a calculator to evaluate the integral. The average population of Nevada between 2010 and 2020 is predicted to be about 2.891 million people.